Oscillator neural network model with distributed native frequencies

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Abstract.   We study the associative memory of an oscillator neural network with distributed native frequencies. The model is based on the use of the Hebb learning rule with random patterns ($\xi_i^\mu = \pm 1$), and the distribution function of native frequencies is assumed to be symmetric with respect to its average. Although the system with an extensive number of stored patterns is not allowed to become entirely synchronized, long time behaviours of the macroscopic order parameters describing partial synchronization phenomena can be obtained by discarding the contribution from the desynchronized part of the system. The oscillator network is shown to work as associative memory accompanied by synchronized oscillations. A phase diagram representing properties of memory retrieval is presented in terms of the parameters characterizing the native frequency distribution. Our analytical calculations based on the self-consistent signal-to-noise analysis are shown to be in excellent agreement with numerical simulations, confirming the validity of our theoretical treatment.

1. Introduction

Most of the theoretical models for associative memory of neural networks as typified by the Hopfield model [1] have been based on the idea of rate coding, which assumes that information is coded in the firing rate of a neuron at a particular time. On the other hand, some experimental results on the visual cortex have been suggesting the possibility of another scheme of information coding, that is, the concept of temporal coding which assumes that information is coded in the relative timing of the firing pulses [2].

Stimulated by these experimental findings, studies of the temporal coding are becoming an active area of theoretical brain researches. Recently, there have been an increasing number of papers reporting neurophysiological experimental findings of synchronization phenomena of neurons in animal's brains [3]. Synchronized firings of neurons can be considered to play a key role in certain types of information processing such as the binding problem [4].

In order to gain insights into basic properties of the scheme of the temporal coding it will be necessary to study neural network models based on neurons that can be expressed by such nonlinear oscillators as limit-cycle-type and integrate-and-fire-type oscillators. Indeed, a network of integrate-and-fire neurons, where one can be directly concerned with the spike timing of neurons, is one such model and was previously studied [5] to confirm its functioning as associative memory accompanied by synchronized firings.

The simplest theoretical model of synchronization of coupled oscillators will be the case of a system of phase oscillators of Kuramoto *et al* [6–8], which is derived under the assumption of weak coupling limit for general types of limit-cycle oscillators. In an associative memory model based on such a system with the Hebb-type learning rule, the distribution of native

frequencies will become crucial for the behaviour of synchronization or desynchronization of the system [6, 9–11]. If all of the oscillators have an identical native frequency and the couplings are symmetric, the network can be reduced, by an appropriate coordinate transformation, to a system that has a Lyapunov function ensuring stability of its equilibrium state(s), which implies that the network eventually settles into a synchronized state. On the other hand, in the case when native frequencies are distributed, Lyapunov functions in general no longer exist except for a particular case [12] and the network becomes essentially of a dynamic nature; one may expect partial synchronization or more complex oscillatory behaviour.

In this paper, we report results of our detailed analysis of associative memory of an oscillator neural network model with distributed native frequencies. Whereas oscillator networks with a Lyapunov function were studied previously by several authors [13, 14] satisfactory analysis of the case without a Lyapunov function has been conducted far less [15].

2. Analysis of the model

The oscillator network model we consider is described by the following phase evolution equations [6]:

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \omega_i - \sum_j J_{ij} \sin(\phi_i - \phi_j) \qquad (i = 1, \ldots, N) \tag{1}$$

where N denotes the total number of neurons, and ϕ_i and ω_i respectively denote the phase and the native frequency of the ith neuron. We assume the strength of interactions to be given according to the Hebb learning rule,

$$J_{ij} = \frac{1}{N} \sum_\mu \xi_i^\mu \xi_j^\mu \tag{2}$$

where $\{\xi_i^\mu\}$ ($\mu = 1, \ldots, p$) represents the μth stored patterns. We consider the case when the number of patterns is extensive, $p = \alpha N$. In order to elucidate the effect of the distribution of native frequencies on the behaviour of synchronization we want to make the model as simple as possible, and then we assume ξ_i^μ to take values ± 1 rather than continuous values with $\xi_i^\mu = \exp(\mathrm{i}\theta_i^\mu)$, ($\theta_i^\mu = [0 : 2\pi]$), which would be more appropriate for the study of temporal coding itself.

Furthermore, we deal with a simple case where the native frequency distribution $p(\omega)$ is discrete and symmetric with respect to the central frequency ω_0: $p(\omega) = \sum_{k=-L}^{L} C_k \delta(\omega - \omega_k)$, $\omega_k + \omega_{-k} = 2\omega_0$, $C_k = C_{-k} \geqslant 0$, $\sum_{k=-L}^{L} C_k = 1$. It is noted that without loss of generality one can set $\omega_0 = 0$ as a result of rotational symmetry of the phase evolution equation (1). If the native frequencies are the same, i.e., $\omega_i \equiv \Omega$ independent of i, all of the oscillators can become synchronized with $\phi_i(t) = \phi_i^0 + \Omega t$. In the case of distributed native frequencies, there remain, in general, a group of desynchronized oscillators. Then the total system cannot settle into entirely equilibrium states.

Assuming that influences exerted by a group of desynchronized neurons on the macroscopic behaviour of the total system can be neglected, we use the self-consistent signal-to-noise analysis (SCSNA) [16] to analytically obtain the macroscopic order parameter equations for the time-stationary states of synchronized oscillations of the network accompanying memory retrieval.

Introducing the complex variable $x_k \equiv \exp(\mathrm{i}\phi_k)$ to express the state of the kth neuron, we formally obtain the fixed-point equations by putting $\dot{x}_i = 0$;

$$x_i = \frac{\mathrm{i}\omega_i + \sqrt{|h_i|^2 - \omega_i^2}}{\tilde{h}_i} \qquad |h_i| > \omega_i \tag{3}$$

where $h_i \equiv \sum_j J_{ij} x_j$ is the local field and $\tilde{h}_i$ denotes its complex conjugate. Defining the overlap

$$m^\mu = \frac{1}{N} \sum_i \xi_i^\mu x_i \tag{4}$$

one can rewrite the local field as $h_i = \sum_\mu \xi_i^\mu m^\mu$.

Within the framework of the SCSNA, considering the retrieval solutions of equation (3) for which $m^1 = m \sim O(1)$ and $m^\mu \sim O(1/\sqrt{N})(\mu \neq 1)$, we assume the local field to be described as [16]

$$h_i = \xi_i^1 m + z_i + \Gamma_1 x_i + \Gamma_2 \tilde{x}_i. \tag{5}$$

Here the first term involving m is the signal part, while the remaining terms represent the noise part involving complex Gaussian noise $z_i = u_i + iv_i$ (u_i, v_i, real) together with the effective self-coupling terms proportional to x_i and its complex conjugate $\tilde{x}_i$. We note that equation (3) yields no solutions if $|h_i| < \omega_i$, which does not ensure $|x_i| = 1$. This means that neurons with $|h_i| < \omega_i$ cannot take part in synchronized motions exhibited by neurons with $|h_i| > \omega_i$. Although the desynchronized neurons, each of which is expected to oscillate with a certain modified frequency, will make the local fields be time-dependent quantities, their effect can be expected to cancel out to good approximation provided taking the time average is considered. In a further analysis, we discard the contribution from the desynchronized neurons by setting $x_i = 0$ for $|h_i| < \omega_i$.

Following the standard procedure of the SCSNA, we obtain the self-consistent equations for macroscopic variables in the limit $N \to \infty$. Making use of the rotational symmetry of the phase evolution equation (1) to choose a gauge such that overlap m becomes real i.e. $[\int DuDv\xi^1 \sin\phi]_{\{\omega,\xi^1\}} = 0$, we have

$$m = \frac{1}{N} \sum_i \xi_i^1 x_i(\xi_i^1, \omega_i, z_i, \tilde{z}_i) = \left[\int DuDv\xi^1 \cos\phi \right]_{\{\omega,\xi^1\}} \tag{6a}$$

$$q = \frac{1}{N} \sum_i (\mathrm{Re}\,[x_i])^2 = \left[\int DuDv(\cos\phi)^2 \right]_{\{\omega,\xi^1\}} \tag{6b}$$

$$U_1 = \frac{1}{N} \sum_i \frac{\partial x_i}{\partial z_i} = \left[\int \frac{DuDv}{2\{(\xi^1 m + u)\cos\phi + v\sin\phi\} + 4\Gamma_2 \cos 2\phi} \right]_{\{\omega,\xi^1\}} \tag{6c}$$

$$U_2 = \frac{1}{N} \sum_i \frac{\partial x_i}{\partial \tilde{z}_i} = \left[\int \frac{-\cos 2\phi\,DuDv}{2\{(\xi^1 m + u)\cos\phi + v\sin\phi\} + 4\Gamma_2 \cos 2\phi} \right]_{\{\omega,\xi^1\}} \tag{6d}$$

$$\Gamma_1 = \frac{\alpha(1 - U_1)}{(1 - U_1)^2 - U_2^2} \tag{6e}$$

$$\Gamma_2 = \frac{\alpha U_2}{(1 - U_1)^2 - U_2^2} \tag{6f}$$

$$Q_1 = \frac{\alpha q}{(1 - U_1 - U_2)^2} \tag{6g}$$

$$Q_2 = \frac{\alpha(1 - q)}{(1 - U_1 + U_2)^2} \tag{6h}$$

$$DuDv = \frac{dudv}{2\pi\sqrt{Q_1 Q_2}} \exp\left[-\frac{1}{2}\left(\frac{u^2}{Q_1} + \frac{v^2}{Q_2} \right) \right] \tag{6i}$$

with $\phi(\xi^1, \omega, u, v)$ being implicitly determined by

$$f(\phi) \equiv -\omega + (\xi^1 m + u)\sin\phi - v\cos\phi + \Gamma_2 \sin 2\phi = 0. \tag{7}$$

In the above equations, $[\ ]_{\{\omega,\xi^1\}}$ means taking the average over the distribution $p(\omega)$ and the pattern $\{\xi^1\}$, and the Gaussian integration $DuDv$ is to be performed over the noise u, v satisfying the condition $|h_i| > \omega_i$. It is noted that in performing the Gaussian integration in equations (6a)–(6i), one has to take into account the Maxwell rule to pick up the relevant solution ϕ_i, when equation (7) admits multi-solutions owing to the presence of the self-coupling term (Γ_2 term). Unlike the Γ_2 term, the Γ_1 term has no contribution to the equilibrium fixed-point equation (7).

In what follows, for the sake of simplicity we deal mainly with the case

$$p(\omega) = C_0\delta(\omega) + \frac{1-C_0}{2}\delta(\omega - \omega_1) + \frac{1-C_0}{2}\delta(\omega + \omega_1) \qquad (L = 1, \omega_0 = 0). \qquad (8)$$

A characteristic feature of the distribution is that the presence of oscillators with central frequency ω_0 is allowed with a finite fraction C_0 and the effect of the desynchronized part can be described in terms of ω_1 and C_0 alone.

Setting $\omega_1 = 0$ recovers the case that allows an energy function that is bounded from below, $E[\{\phi_i\}] \equiv -\frac{1}{2}\sum_{ij} J_{ij}\cos(\phi_i - \phi_j)$.

Then all of the oscillators get synchronized for large times with the equilibrium configuration $\{\phi_i\}$ determined by equations (3) or (7) together with the Maxwell rule, which is explained in figure 1. Analysis based on the SCSNA of such networks as having Lyapunov functions is on the same level of approximation as the replica symmetric calculations [16–18]. Figure 2 represents the dependence of m on the loading rate α computed from the SCSNA equations (6) and (7). We see that the storage capacity is given by $\alpha_c = 0.0395$ with $m_c = 0.68$. The present value of the storage capacity slightly differs from the result reported previously [14], which was obtained by an inappropriate treatment of the Maxwell rule [19]. We note that the Maxwell rule becomes necessary only for α in the neighbourhood of α_c. Under the assumption of random patterns with $\xi_i^\mu = \pm 1$, the magnitude of Γ_2 plays a crucial role in determining the α_c, as can be seen in figure 2. This should be compared with the case of $\xi_i^\mu = \exp(i\theta_i^\mu)$ ($\theta_i^\mu = [0 : 2\pi]$), where the order parameter equations involve no Γ_2 term as in the case of Cook's model [13] (see also the recent work in [15]). We note here that when discrete patterns with Q-state variables $\theta_i^\mu = 2\pi n_i^\mu/Q$, $(n_i^\mu = 0, \ldots, Q-1)$ are considered, the Γ_2 term does not appear in the order parameter equations, if $Q \geqslant 3$.

The effect of the distribution of native frequencies on the behaviour of memory retrieval is summarized in figure 3, where overlap m from the SCSNA equations is plotted as a function of α and ω_1 for $C_0 = 0.7$. There appear two distinct retrieval regimes separated by a valley or gap located around a region with an intermediate value of ω_1 in the m–α–ω_1 space. In the regime with small ω_1, most of the oscillators undergo synchronized motions. The storage capacities α_c corresponding to the edge of the m surface are observed to decrease, as ω_1 is increased, to attain a certain minimum value, where a crossover to the other regime occurs. In the regime with large ω_1, oscillators with ω_1 oscillate with their own frequencies that are modified from the original ω_1 as a result of entrainment phenomenon. Such a behaviour can easily be understood by considering the system in the limit $\omega_1 \to \infty$. In the large ω_1 limit, while most of the oscillators with $\omega_i = \omega_0$ become synchronized, the oscillators with $\omega_i = \pm\omega_1$ cannot become synchronized and they oscillate with their own frequency $\pm\omega_1$ independently of the synchronized neurons with $\omega_i = \omega_0$. It is clearly seen in this case that the desynchronized neurons do not contribute to the time-averaged local fields acting on the synchronized neurons. Then the system can be viewed as a diluted system with only a fraction C_0 of neurons participating in memory retrieval. It is interesting to note that the storage capacity α_c in the large ω_1 region increases as ω_1 is increased. The crossover between the two regimes or the gap in the $m(\omega_1)$ curve with fixed α can be more clearly seen in figure 4, which displays the ω_1 dependence of overlap m obtained from the SCSNA together with the result
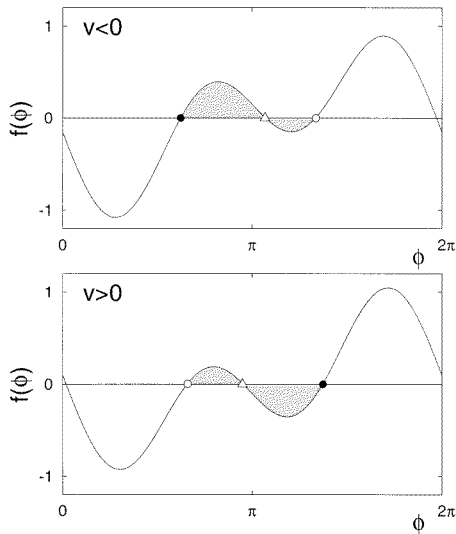
Figure 1. Maxwell rule used in the SCSNA for picking up the relevant solution for output x_i as a function of u and v among multi-solutions to equation (7). In a graphical representation of solving equation (7) for which three solutions (given by the intersections of the two curves) appear as u or v is varied, the relevant solution is chosen as the external intersection that delimits the larger area enclosed between the two curves. Such a solution is marked by the filled circle. The Maxwell rule ensures the condition of the free energy minimum [17] in a system with a Lyapunov function as in the case of the liquid–vapour phase transition.
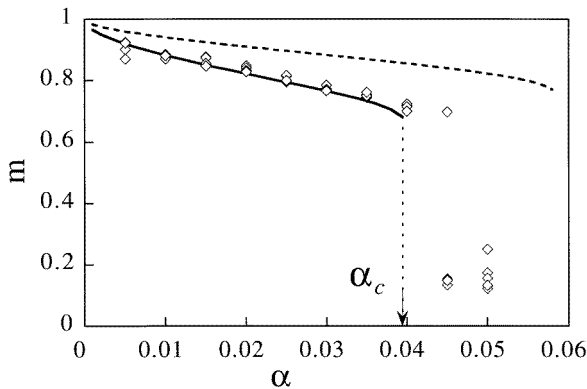

Figure 2. α dependence of the overlap m obtained from the SCSNA (solid curve) together with that from numerical simulations with $N = 4000$ in the case $p(\omega) = \delta(\omega - \omega_0)$. To observe the result that the contribution of the Γ_2 term in equation (5) or (7) to the value of m is significant, we plot values of m obtained by deliberately setting $\Gamma_2 = 0$ (broken curve).

of numerical simulations ($N = 2000$) for a fixed value of α.

As the fraction C_0 of the oscillators with the central frequency ω_0 is varied, three types of behaviour of the $m(\omega_1)$ curve showing the crossover between the two regimes occur. In figure 5, we show the ω_1 dependence of the overlap m obtained for $C_0 = 0.5$, 0.7 and 0.9 with $\alpha = 0.02$. While for a large value of C_0 one sees a continuous crossover between the small ω_1 regime and the large ω_1 regime, for a small value of C_0 the large ω_1 regime disappears. Only for intermediate values of C_0, does one observe the gap in the $m(\omega_1)$ curve appearing.
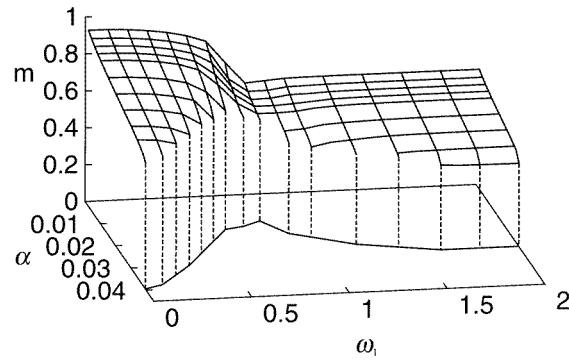
Figure 3. Phase diagram showing m-surface plotted as a function of α and ω_1 for the network with distributed native frequencies with $p(\omega) = 0.7\delta(\omega) + 0.15\delta(\omega - \omega_1) + 0.15\delta(\omega + \omega_1)$. We observe a valley or gap separating the small ω_1 regime from the large ω_1 one. The dependence of the storage capacities α_c on ω_1 is represented by the projected curve on the α–ω_1 plane.
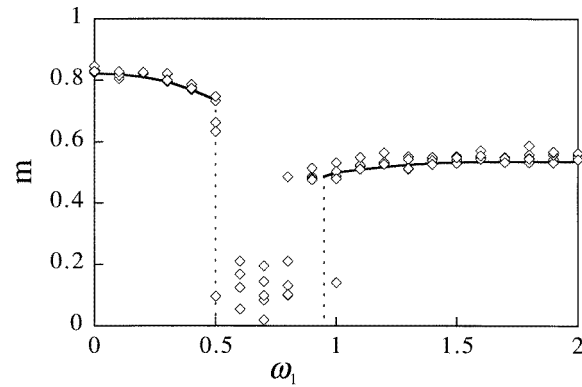

Figure 4. Dependence of m on ω_1 for the network with $\alpha = 0.02$ and $C_0 = 0.7$ obtained from the SCSNA (curve) and numerical simulations with $N = 2000$ (dots). The gap separating the two regimes with different types of synchronization is clearly depicted and is in excellent agreement with the results of numerical simulations.
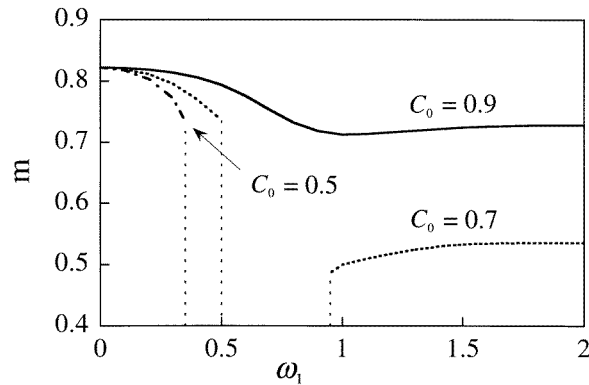

Figure 5. Dependence of m on ω_1 for $\alpha = 0.02$ and $C_0 = 0.5, 0.7, 0.9$ obtained from the SCSNA.
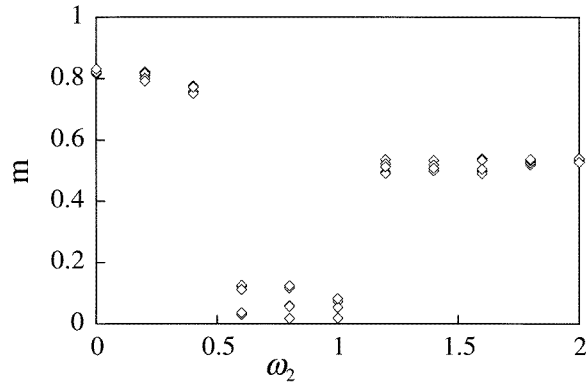
Figure 6. The numerical simulation result ($N = 4000$) for $L = 2$ showing the ω_2 dependence of m in the case with $\omega_0 = 0$, $C_0 = 0$, $\omega_1 = 0.1$, $C_1 = 0.35$ and $\alpha = 0.02$ (see equation (9)).

The occurrence of such two retrieval regimes is not restricted to the simplest case of $L = 1$, but is also observed for the more general case with $L \geqslant 2$. When a sufficiently large number of oscillators have native frequencies near the central one ω_0, the system can behave in the same way as that of $L = 1$ with $C_0 \neq 0$, as is shown figure 6, where the ω_2 dependence of the overlap m obtained by numerical simulations is displayed for the native frequency distribution with $L = 2$ and $C_0 = 0$:

$$p(\omega) = C_1\delta(\omega - \omega_1) + C_1\delta(\omega + \omega_1) + \frac{1 - 2C_1}{2}\delta(\omega - \omega_2) + \frac{1 - 2C_1}{2}\delta(\omega + \omega_2).$$

(9)

Indeed we see a gap separating the small ω_2 regime from the large ω_2 one. Although there are no oscillators with $\omega_0 = 0$ because of $C_0 = 0$, most of the oscillators become synchronized for the small ω_2 regime.

3. Summary and discussions

We have studied the behaviour of the oscillator neural network system with distributed native frequencies. To date, analysis of associative memory models has been mostly confined to the case of networks with an energy or Lyapunov function that allows the use of replica symmetric calculations of statistical mechanics. In this paper, making use of the method of the SCSNA that is free from the energy concept [6, 20] we have succeeded in analysing properties of *the associative memory accompanied by synchronized oscillations* in the prototype oscillator network that has no Lyapunov functions except for the case when $p(\omega) = \delta(\omega - \omega_0)$. *We have shown that the oscillator network can work as associative memory based on temporal coding of a simple type even in the presence of distribution of native frequencies.* Our approach has taken advantage of the fact that such temporal attractors as limit cycles of certain types of dynamical systems can be reduced to a fixed-point-type attractors as a result of the system's symmetry property.

The distribution of native frequencies does not allow the present coupled oscillator system to settle into an entirely synchronized state but into a partially synchronized one. The contributions from desynchronized neurons to the macroscopic behaviour of the system have, however, been found to be almost negligible. Thereby, the partially synchronized state of the system has turned out to be almost determined by the long time behaviour of the group of synchronized neurons, which can be described by fixed-point-type attractors giving rise to

retrieval states. In other words, *memory retrieval is achieved by synchronization of oscillatory motions of neurons.*

Under the assumed type of native frequency distribution we have found that *the partial synchronization is classified into a high degree of synchronization that occurs for small ω_1 with overlap m large and a low degree of synchronization that occurs for large ω_1 with m small.*

Finally, we note that oscillator neural networks are considered to have advantage over fixed-point-type neural networks in several respects. First, oscillator neural networks exhibit the ability to easily and efficiently discriminate a successful retrieval from an unsuccessful one, because the settling into a retrieval state of oscillator neural networks implies the appearance of oscillations with an appreciable amplitude and the central frequency ω_0 in the overlap, and hence, the local fields of neurons.

Second, by utilizing phase as well as amplitude as dynamical variables representing output of a neuron [21] it becomes possible for neural information to be processed in terms of spatio-temporal patterns of neuronal firings. In particular, information on the time domain is available by employing the scheme of temporal coding [11, 13, 15, 22] where assuming uniformly distributed random numbers θ_i^μ on $[0, 2\pi)$ for components of the memory patterns, the phase difference $\phi_i - \phi_j$ between the two oscillators i and j eventually settles into the difference $\theta_i^\mu - \theta_j^\mu$ of the memory pattern μ. Our present model setting based on the use of binary patterns with $\theta_i^\mu = 0, \pi$ [14] presents a special as well as simple case of the above temporal coding, where the pattern of synchronization is either of in phase or out of phase. Such a case may be related to the problem of segmentation of an object from its background [23–25].

More generally, pattern segmentation [26] seems to be one of the unique features of oscillator neural networks as has been studied by Wang *et al* [27], who pointed out that the use of limit cycles as attractor states of associative memory facilitates switching between one pattern and another on the time domain.

While studying the important issue of the functionality of pattern segmentation provided by oscillator neural networks will require taking into account some specific ingredients such as the sparseness of the memory patterns and appropriate inhibitory couplings, our simple model will have wide applicability in exploring the computational ability or relevance exhibited by oscillator neural networks from the viewpoint of analytical studies. Extending the assumed symmetric native frequency distribution to more general cases of asymmetric ones as well as continuous ones is straightforward. Results of such issues together with details of this paper will be reported elsewhere.

References

[1] Hopfield J J 1982 *Proc. Natl Acad. Sci. USA* **79** 2554
[2] Gray C M, König P, Engel A K and Singer W 1989 *Nature* **338** 334
[3] Vaadia E, Haalman I, Abeles M, Bergman H, Prut Y, Slovin H and Aertsen A 1995 *Nature* **373** 515
[4] Damasio A R 1990 *Semin. Neurosci.* **2** 287
[5] Yoshioka M and Shiino M 1998 *Phys. Rev.* E **58** 3628
[6] Kuramoto Y 1984 *Chemical Oscillations, Waves, and Turbulence* (Berlin: Springer)
[7] Daido H 1990 *J. Stat. Phys.* **60** 753
[8] Sakaguchi H and Kuramoto Y 1986 *Prog. Theor. Phys.* **76** 576
[9] Shiino M and Frankowicz M 1989 *Phys. Lett.* A **136** 103
[10] Kuramoto Y, Aoyagi T, Nishikawa I, Chawanya T and Okuda K 1992 *Prog. Theor. Phys.* **87** 1119
[11] Arenas A and Pérez Vicente C J 1994 *Europhys. Lett.* **26** 79
[12] van Hemmen J L and Wreszinski W F 1993 *J. Stat. Phys.* **72** 145
[13] Cook J 1989 *J. Phys. A: Math. Gen.* **22** 2057
[14] Aonishi T 1998 *Phys. Rev.* E **58** 4865
[15] Aonishi T, Kurata K and Okada M 1999 *Phys. Rev. Lett.* **82** 2800

[16] Shiino M and Fukai T 1992 *J. Phys. A: Math. Gen.* **25** L375
    Shiino M and Fukai T 1993 *Phys. Rev.* E **48** 867
[17] Shiino M and Fukai T 1990 *J. Phys. A: Math. Gen.* **23** L1009
[18] Kühn R and Bös S 1993 *J. Phys. A: Math. Gen.* **26** 831
[19] Fukai T and Shiino M 1995 *Neural Comput.* **7** 529
[20] Yoshioka M and Shiino M 1997 *Phys. Rev.* E **55** 7401
[21] Aoyagi T 1995 *Phys. Rev. Lett.* **74** 4075
[22] Aoyagi T and Kitano K 1997 *Phys. Rev.* E **55** 7424
[23] von der Malsburg C and Schneider W 1986 *Biol. Cybern.* **54** 29
[24] von der Malsburg C and Buhmann J 1992 *Biol. Cybern.* **67** 233
[25] Terman D and Wang D L 1995 *Physica* D **81** 148
[26] Ritz R, Gerstner W and van Hemmen J L 1994 Associative binding and segregation in a network of spiking
    neurons *Models of Neural Networks II* ed E Domany, J L van Hemmen and K Schulten (Berlin: Springer)
    ch 5
[27] Wang D L, Buhmann J and von der Malsburg C 1990 *Neural Comput.* **2** 94